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ON STANDARD INTEGRAL TABLE ALGEBRAS GENERATED BY AN ELEMENT OF WIDTH 2

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ABSTRACT

The paper studies standard integral table algebras containing an element of width two. It is shown that the table subalgebra generated by such an element is exactly isomorphic to one of the table algebras described in Main Theorems 1.1 and 1.2.

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1. Introduction

Definition 1.1 ([ABEF], [B1], [B2]): A finite dimensional, associative and commutative complex algebra A with identity 1 is called a **table algebra** with respect to a distinguished basis $\mathbf{B}$ (and $\mathbf{B}$ is its **table basis**) if and only if the following hold:

I) $1 \in \mathbf{B}$.

II) The structure constants of A with respect to the basis $\mathbf{B}$ are non-negative real numbers:

$$ab = \sum_{c \in \mathbf{B}} \lambda_{abc} c, \quad \text{for some } \lambda_{abc} \in \mathbb{R}_{\geq 0}.$$

III) There is an algebra (anti)automorphism (denoted by $\bar{\ }$) of A such that $\bar{\bar{a}} = a$ for all $a \in A$ and $\overline{\mathbf{B}} = \mathbf{B}$ (an element $b \in \mathbf{B}$ is called **real** if $\bar{b} = b$).

IV) For all $a, b \in \mathbf{B}$, $\lambda_{ab1} = 0$ if $a \neq \bar{b}$ and $\lambda_{a\bar{a}1} > 0$.

By [AB, Lemma 2.9], there is a unique algebra homomorphism $|\ |: A \to \mathbb{C}$ such that $|b| = |\bar{b}| \in \mathbb{R}^+$ for all $b \in \mathbf{B}$. The positive real numbers $|b|$, $b \in \mathbf{B}$ are called the **degrees of** $(A, \mathbf{B})$. In what follows we also denote $\mathbf{B}^\# = \mathbf{B} \setminus \{1\}$, $\mathbf{S}^+ := \sum_{s \in \mathbf{S}} s$ and $|\mathbf{S}| = \sum_{s \in \mathbf{S}} |s|$.

Table algebras are closely related to C-algebras introduced by Kawada [Kaw] and Hoheisel [H]. The axioms of C-algebra may be obtained from I–IV if we replace the condition $\lambda_{abc} \in \mathbb{R}_{\geq 0}$ by $\lambda_{abc} \in \mathbb{R}$ and require the mapping $\sum_{b \in \mathbf{B}} x_b b \mapsto \sum_{b \in \mathbf{B}} x_b \lambda_{b\bar{b}1}$ to be an algebra homomorphism.

Definition 1.2 ([B1]): An **integral table algebra** (abbreviated ITA) is a table algebra $(A, \mathbf{B})$ such that all the structure constants λ_{abc} and all the degrees $|b|$ are rational integers.

Any finite group G yields two examples of ITA's: $(Z(\mathbb{C}G), \mathrm{Cla}(G))$, the center of the group algebra, with table basis the set of sums $\hat{C}$ of G-conjugacy classes C, with automorphism $\bar{\ }$ extended linearly from inversion in G, and with degrees $|\hat{C}| = |C|$ for all $\hat{C} \in \mathrm{Cla}(G)$; and $(\mathrm{Ch}(G), \mathrm{Irr}(G))$, the ring of complex valued class functions on G, with table basis the set of irreducible characters of G, with automorphism $\bar{\ }$ extended linearly from complex conjugation of characters, and with degrees $|\chi| = \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$. Another example is the Bose–Mesner algebra of a commutative association scheme, with table basis the set of adjacency matrices, [BI, Section II.2].

Definition 1.3 ([BX1]): A table algebra $(A, \mathbf{B})$ is called **homogeneous (of degree λ)** iff $|\mathbf{B}| > 1$ and, for some fixed positive real number λ,

$$|b| = \lambda \quad \text{for all } b \in \mathbf{B}^{\#};$$

a homogeneous integral table algebra is denoted by HITA.

Any table algebra may be **rescaled** (replacing each table basis element by a positive scalar multiple) to one which is homogeneous, and any ITA can be rescaled to a homogeneous ITA [BX1, Theorem 1].

Let $(A, \mathbf{B})$ be a table algebra. For all $x \in A$, $\mathsf{Supp}_{\mathbf{B}}(x) = \mathsf{Supp}(x)$ denotes the collection of all elements of $\mathbf{B}$ which appear with nonzero coefficient in the decomposition of x. A nonempty subset $\mathbf{C} \subseteq \mathbf{B}$ is called a **table subset of B** (notation $\mathbf{C} \leq \mathbf{B}$) iff $\mathsf{Supp}(ab) \subseteq \mathbf{C}$ for all $a, b \in \mathbf{C}$. Any table subset is stable under $\bar{}$ and contains 1_A, [AB, Proposition 2.7]; [BI, Proposition 2.13]. For any $c \in \mathbf{B}$, the set $\mathbf{B}_c$ defined by

$$\mathbf{B}_c := \bigcup_{n=1}^{\infty} \mathsf{Supp}(c^n)$$

is easily seen to be a table subset of $\mathbf{B}$, called the **table subset generated by** c.

An element c of $\mathbf{B}$ is called **faithful** iff $\mathbf{B}_c = \mathbf{B}$. Let G be any finite group. Then $\hat{C} \in \mathrm{Cla}(G)$ is faithful iff $\langle C \rangle = G$, and $\chi \in \mathrm{Irr}(G)$ is faithful iff χ is faithful in the usual character-theoretic sense.

Definition 1.4 ([B5], [BX2]): An element $b \in \mathbf{B}$ is called **standard** iff $|b| = \lambda_{b\bar{b}1}$. A table algebra $(A, \mathbf{B})$ is **standard** if every element of $\mathbf{B}$ is standard.

In what follows we use abbreviations S(I)TA and H(I)TA for standard (integral) and homogeneous (integral) table algebras, respectively.

Definition 1.5 ([BX2]): Two table algebras $(A, \mathbf{B})$ and $(A', \mathbf{B}')$ are called **isomorphic** (denoted $\mathbf{B} \simeq \mathbf{B}'$) when there exists an algebra isomorphism $\psi: A \to A'$ such that $\psi(\mathbf{B})$ is a rescaling of $\mathbf{B}'$; and the algebras are called **exactly isomorphic** (denoted $\mathbf{B} \simeq_x \mathbf{B}'$) when $\psi(\mathbf{B}) = \mathbf{B}'$. So $\mathbf{B} \simeq_x \mathbf{B}'$ means that $\mathbf{B}$ and $\mathbf{B}'$ yield the same structure constants.

Definition 1.6: Let $(A, \mathbf{B})$ be a table algebra. Then $c \in \mathbf{B}$ is called **linear** iff $\mathsf{Supp}(c^n) = \{1\}$ for some $n > 0$. This is equivalent to $\mathsf{Supp}(c\bar{c}) = \{1\}$, [AB, Proposition 4.2]. A table subset is called **abelian** iff each of its elements is linear. The set of all linear elements of $\mathbf{B}$, denoted by $\mathbf{L}(\mathbf{B})$, is a table subset [AB, Proposition 4.2].

Definition 1.7: Let $(A, \mathbf{B})$, $(C, \mathbf{D})$ be two table algebras. Then one can define their **tensor product** $(A \otimes_{\mathbb{C}} C, \mathbf{B} \otimes \mathbf{D})$ where the distinguished basis $\mathbf{B} \otimes \mathbf{D}$ is the set of tensors $b \otimes d$, $b \in \mathbf{B}$, $d \in \mathbf{D}$. If both $(A, \mathbf{B})$ and $(C, \mathbf{D})$ are standard, then so is their tensor product.

If $(C, \mathbf{D})$ is standard then $(A \otimes_{\mathbb{C}} C, \mathbf{B} \otimes \mathbf{D})$ contains a subalgebra $(A, \mathbf{B}) \wr (C, \mathbf{D})$ spanned, as a vector space, by the following basis:

$$\mathbf{B} \wr \mathbf{D} := \{1 \otimes d \colon d \in \mathbf{D}\} \cup \{b \otimes \mathbf{D}^{+} \colon b \in \mathbf{B}^{\#}\}.$$

A direct check shows that a subspace spanned by the above basis is a subalgebra of $(A \otimes_{\mathbb{C}} C, \mathbf{B} \otimes \mathbf{D})$ that satisfies all the axioms. In what follows we shall denote it by $(A \wr C, \mathbf{B} \wr \mathbf{D})$ and call it a **wreath product** of $(C, \mathbf{D})$ by $(A, \mathbf{B})$. The dimension of $(A \wr C, \mathbf{B} \wr \mathbf{D})$ is always equal to $\dim(A) + \dim(C) - 1$.

Example 1.8: In [BX1], Blau and Xu defined three families of homogeneous table algebras of degree k, as follows:

1. $\mathbf{T}_0(k) := \{1, b\}$ with $b^2 = k \cdot 1 + (k-1)b$.
2. $\mathbf{V}(3, k) := \{1, b, c\}$ with

$$b^2 = k \cdot 1 + (k-1)c,$$
$$bc = (k-1)b + c,$$
$$c^2 = k \cdot 1 + b + (k-2)c.$$

3. $\mathbf{V}(2, k, \mu) := \{1, b\}$ with $b^2 = k\mu \cdot 1 + (k - \mu)b$ where $0 < \mu \le k$, $\mu, k \in \mathbb{N}$.

In what follows we set $\mathbf{V}_2 := \mathbf{V}(2,3,2)$, $\mathbf{V}_3 := \mathbf{V}(3,3)$.

Definition 1.9 ([BX1])*:* Let $(A, \mathbf{B})$ be a table algebra and $b \in \mathbf{B}$. The **width** of b is defined as $|\mathsf{Supp}(b\bar{b})|$. The width of $(A, \mathbf{B})$ is defined as a maximal value of $|\mathsf{Supp}(b\bar{b})|, b \in \mathbf{B}$.

Integral table algebras of width 1 were classified in [AB] and they are abelian, up to rescaling, the group algebras of abelian groups. The first step in the study of table algebras of width two was done by Blau and Xu. To formulate their result we need the following notation. Let $(A, \mathbf{B})$ be a table algebra, H an abelian group and λ a positive real parameter. We set

$$(\mathbf{B} \otimes H)' := \{b \otimes h \colon b \in \mathbf{B} \setminus \{1\}, h \in H\} \cup \{1 \otimes \lambda h \colon h \in H \setminus \{1\}\} \cup \{1 \otimes 1\}.$$

THEOREM 1.10 ([BX1, Theorem 3]): *Let $(A, \mathbf{B})$ be an integral table algebra which is homogeneous of degree λ for some $\lambda > 2$, and is such that $\mathbf{B}$ contains a standard, faithful element of width 2. Then $\mathbf{B}$ is exactly isomorphic to*

$(\mathbf{T}_0(\lambda)\otimes\mathbb{Z}_m)'$ or $(\mathbf{V}(3,\lambda)\otimes\mathbb{Z}_m)'$ *for some* $m\in\mathbb{Z}^+$,

$$\mathbf{Z}'_m := \{1\}\cup\{\lambda h\colon h\in\mathbb{Z}_m\setminus\{1\}\}.$$

In [BX2], Blau and Xu proved the following statement.

COROLLARY 1.11 ([BX2, Corollary 2.15]): *Assume that* $(A,\mathbf{B})$ *is a homogeneous ITA of degree* 3, *and that* $\mathbf{B}$ *contains a faithful element* b *of width* 2. *Then* $\mathbf{B}\simeq_x(\mathbf{V}_2\otimes\mathbf{Z}_m)',(\mathbf{V}_3\otimes\mathbb{Z}_m)'$, *or* $(\mathbf{T}_0(3)\otimes\mathbb{Z}_m)'$, *for some* $m\geq 1$. *So if* $\mathbf{B}$ *is standard, then* $\mathbf{B}\simeq_x\mathbf{V}_3$ *or* $\mathbf{T}_0(3)$. *If* b *is real, then* $m\leq 2$.

In [R] Rahnamai Barghi classified ITAs which contain a real standard faithful element of width 2. In this paper we classify SITAs of width two generated by one element.

If $(A,\mathbf{B})$ is a SITA and $a\in\mathbf{B}$ is an arbitrary element of width 2, then $\mathsf{Supp}(a\bar{a})=\{1,b\}$ for some $b\in\mathbf{B}^{\#}$. Two cases are possible: $|b|\leq|a|$ and $|b|>|a|$. These cases are covered by the following two Theorems.

MAIN THEOREM 1.1: *Let* $(A,\mathbf{B})$ *be a SITA. Let* $a\in\mathbf{B}^{\#}$ *be an element of width* 2 *and degree* $k\geq 2$. *If* $|\mathsf{Supp}(a\bar{a})|\leq|a|+1$, *then one of the following holds:*

- (I) $k=2$ *and* $\mathbf{B}_a$ *may be rescaled to a HITA of degree* 2. *(These algebras were classified by Blau in [B2].)*
- (II) $k\geq 3$ *and one of the following holds:*
  - (a) $\mathbf{B}_a\simeq_x\mathbb{Z}_m\otimes\mathbf{T}_0(k)$,
  - (b) $\mathbf{B}_a\simeq_x\mathbb{Z}_m\otimes\mathbf{V}(3,k)$,
  - (c) $a\bar{a}=k\cdot 1+k\cdot b$ *for some* $b\in\mathbf{B}^{\#}$, $|b|=k-1$ *and* $\mathbf{B}\simeq_x\mathbf{B}(\mathbb{Z}_m,d)$ *for some degree function* $d\colon\mathbb{Z}_m\to\mathbb{N}$ *related to* k *(see Subsection 2.1 for the exact definition of algebras of type* $\mathbf{B}(\mathbb{Z}_m,d)$*).*

Remark: The conditions of the Theorem are always satisfied if an element of maximal degree is of width two.

Finite groups related to SITA induced from conjugacy classes satisfying II of our Main Theorem are classified in Proposition 3.7.

The next result resolves the case of $|\mathsf{Supp}(a\bar{a})|>|a|+1$.

MAIN THEOREM 1.2: *Let* $(A,\mathbf{B})$ *be a SITA of width two. If there exist* $a\in\mathbf{B}^{\#}$ *such that* $|a|=k\geq 3$, *and* $|\mathsf{Supp}(a\bar{a})|>|a|+1$, *then* $a\bar{a}=k\cdot 1+lx$ *for some* $l\in\mathbb{N}$, *and* $x\in\mathbf{B}$ *with* $l\mid k(k-1), l<k-1$, *and* $|x|=k(k-1)/l$; *and* $\mathbf{B}_a\simeq_x\mathbf{B}(\mathbb{Z}_m,d)$ *for some degree function* $d\colon\mathbb{Z}_m\to\mathbb{N}$ *related to* $|x|+1$.

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1.1. Known facts and some consequences.

Definition 1.12 ([AF5, Section 2], [B1, Definition 1.17]): The **order of** $(A, \mathbf{B})$ is defined as

$$o(\mathbf{B}) := \sum_{b \in \mathbf{B}} |b|^2 / \lambda_{b\bar{b}1}.$$

Note that in the case of either $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ or $(\mathrm{Ch}(G), \mathrm{Irr}(G))$ for any finite group G, the order of the table algebra equals $|G|$. If $\mathbf{B}$ is the set of adjacency matrices of a commutative association scheme, then $o(\mathbf{B})$ is the cardinality of the underlying set. For any $\mathbf{B}$, $o(\mathbf{B})$ clearly is invariant under rescaling. Since $b \in \mathbf{B}$ is linear iff $\lambda_{b\bar{b}1} = |b|^2$, we immediately have

PROPOSITION 1.13: *If $\mathbf{B}$ is abelian, then $o(\mathbf{B}) = |\mathbf{B}|$.*

Given a table subset $\mathbf{C}$ of a table algebra $(A, \mathbf{B})$, one can define a **quotient** table algebra $\mathbf{B}/\mathbf{C}$ in the following way. By [B3], Corollary 3.13 there exists a unique idempotent, denoted $e_{\mathbf{C}}$, which is a positive real scalar times $\sum_{b \in \mathbf{C}} |b|/\lambda_{b\bar{b}1} b$. So, eA is a subalgebra of A. It follows from [B3], Theorems 1, 2 that the algebra Ae with a distinguished basis

$$\{e\} \cup \{eb \colon b \in \mathbf{B} \setminus \mathbf{C} \text{ and } |b| \leq |b'| \text{ for all } b' \in \mathsf{Supp}(eb)\}$$

is a table algebra. In what follows we denote by $\mathbf{B}/\mathbf{C}$ a standard rescaling of the basis defined above. By [B3], $(Ae, \mathbf{B}/\mathbf{C})$ is a table algebra which is an epimorphic image of A. It is called the quotient algebra of $(A, \mathbf{B})$ with quotient subset $\mathbf{C}$.

THEOREM 1.14 ([B3], Corollary 4.5): *If $\mathbf{X}$ is any table subset of $\mathbf{B}$, then $o(\mathbf{B}) = o(\mathbf{X}) \cdot o(\mathbf{B}/\mathbf{X})$.*

Definition 1.15 ([BX1]): Let $(A, \mathbf{B})$ be any table algebra with table subsets $\mathbf{C}, \mathbf{D}$. Then

$$\mathbf{CD} := \bigcup_{\substack{b_i \in \mathbf{C} \\ b_j \in \mathbf{D}}} \mathsf{Supp}(b_i b_j)$$

is a table subset of $\mathbf{B}$, the smallest one which contains $\mathbf{C}$ and $\mathbf{D}$.

LEMMA 1.16 ([BX2]): *Assume that $(A, \mathbf{B})$ is a table algebra, and $\mathbf{C}, \mathbf{D}$ are table subsets of $\mathbf{B}$ with $\mathbf{CD} = \mathbf{B}$ and $\mathbf{C} \cap \mathbf{D} = \{1\}$. If $|\mathsf{Supp}(cd)| = 1$ for all $c \in \mathbf{C}$, $d \in \mathbf{D}$ (which holds if either $\mathbf{C}$ or $\mathbf{D}$ is abelian), then $\mathbf{B} \simeq \mathbf{C} \otimes \mathbf{D}$.*

Definition 1.17 ([BX2]): Fix $b \in \mathbf{B}$. The **stabilizer of** b in $\mathbf{C}$, denoted $\mathrm{sta}_{\mathbf{C}}\, b$, is defined as

$$\mathrm{sta}_{\mathbf{C}}\, b := \{x \in \mathbf{C} | \mathsf{Supp}(bx) = \{b\}\}.$$

It is clear that $\mathrm{sta}_{\mathbf{C}}\, b$ is always a table subset.

LEMMA 1.18 ([BX2]): *Let $(A, \mathbf{B})$ be a table algebra, $\mathbf{L} = \mathbf{L}(\mathbf{B})$ and $b \in \mathbf{B}$. Then $\mathsf{Supp}(b\bar{b}) \cap \mathbf{L} = \mathrm{sta}_{\mathbf{L}}\, b$.*

Definition 1.19 ([AFM2]): Let $(A, \mathbf{B})$ be a table algebra. If $x = \sum_{b \in \mathbf{B}} x_b b$, $y = \sum_{b \in \mathbf{B}} y_b b$, then the scalar product $\langle x, y \rangle$ defined by the formula $\langle x, y \rangle = \sum_{b \in \mathbf{B}} x_b y_b \lambda_{b\bar{b}1}$ is a symmetric positive definite form on A.

PROPOSITION 1.20 ([AFM2]): *Let $(A, \mathbf{B})$ be a standard table algebra. Then:*

(i) *for all $a, b, c \in A$, $\langle ab, c \rangle = \langle b, \bar{a}c \rangle = \langle a, c\bar{b} \rangle$;*
(ii) *for all $a, b \in \mathbf{B}$, $|a|\,|b| = \sum_{x \in \mathbf{B}} \lambda_{abx}|x|$;*
(iii) *if $\mathbf{B}$ is integral, then for all $a, b \in \mathbf{B}$, $\lambda_{abc}|c|$ is divisible by the least common multiple of $|a|, |b|$.*

The next Lemma was proposed and proven by the referee.

LEMMA 1.21: *If $\mathbf{B}_a$ is a SITA and $|a| = 2$ then $|b| \leq 2$ for all $b \in \mathbf{B}_a$.*

Proof: Since $\langle a^2, a^2 \rangle = \langle a\bar{a}, a\bar{a} \rangle \geq 6$, a^2 has a linear constituent u. Suppose first that $\langle a\bar{a}, a\bar{a} \rangle = 6$. Then $\langle a, \bar{a}u \rangle = \langle a^2, u \rangle \geq 0$ implies that $a = \bar{a}u$. Assume by induction on m that for all $2 \leq j \leq m$, all elements of $\mathsf{Supp}(a^j)$ have degree at most 2. Let $x \in \mathsf{Supp}(a^m)$. Then $|x| \leq 2$, and $\langle \bar{a}x, a^{m-1} \rangle = \langle x, a^m \rangle \geq 0$, so that $\bar{a}x$ has a constituent of degree at most 2. Hence, so does $u\bar{a}x = ax$. Then $|ax| \leq 4$ implies that all constituents of ax have degree at most 2. Since x is arbitrary in $\mathsf{Supp}(a^m)$, the same holds for all constituents of $aa^m = a^{m+1}$. The result follows in this case. Now suppose that $\langle a\bar{a}, a\bar{a} \rangle > 6$. Then $a\bar{a} = 2(1 + v)$ for some linear element v. It follows that $\mathrm{sta}\, a = \{1,\ v\}$, and that $\mathbf{B}_a/\{1,\ v\}$ is abelian. Hence for all $c \in \mathbf{B}_a$, $c\bar{c} = |c| \cdot 1 + |c|(|c| - 1)v$. Then $\langle cv, c \rangle = \langle c\bar{c}, v \rangle = |c|(|c| - 1)$ implies that the coefficient of c in cv is $|c| - 1$. But $|cv| = |c|$, so that $|c| \leq 2$. ∎

2. The algebra $\mathbf{B}(G,d)$ and its characterizations

2.1. THE ALGEBRA $\mathbf{B}(G,d)$. Let G be a finite abelian group written multiplicatively with identity 1. Fix an arbitrary $2 < k \in \mathbb{R}$. A partial function $d\colon G \to \mathbb{R}$ is called a **degree function related to** k if it satisfies the following conditions:

(D1) The domain of d is a subgroup of G denoted as $G^{<k}$;

(D2) $d(1) = 1$;

(D3) $d(g) = d(g^{-1}) \in [1, k-1)$ for each $g \in G^{<k}$.

The function d is called **integral** if $\mathrm{Im}(d) \subseteq \mathbb{Z}$.

We extend the domain of d to the whole group G by setting $d(g) := k$ for $g \in G \setminus G^{<k}$. We also set

$$d^*(g) := k - d(g), \varphi(g) := \frac{d(g)d^*(g)}{(k-1)}.$$

It is easy to see that $g \in G^{<k} \iff \varphi(g) \neq 0$.

In the algebra $\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}]$ we define the basis $\mathbf{B}(G,d)$ as follows:

$$\mathbf{B}(G,d) := \{b_g\}_{g\in G} \cup \{b_g^*\}_{g\in G^{<k}},$$

where

$$b_g := (d(g)g, \sqrt{\varphi(g)}g),\ g \in G \quad \text{and} \quad b_g^* := (d^*(g)g, -\sqrt{\varphi(g)}g),\ g \in G^{<k}.$$

If $\varphi(g) \neq 0$, then

$$(1) \qquad (g,0) = \frac{b_g + b_g^*}{k}; \quad (0,g) = \frac{d^*(g)b_g - d(g)b_g^*}{k\sqrt{\varphi(g)}};$$

note that $(g,0) = (b_g + b_g^*)/k$ even if $\varphi(g) = 0$.

Since $b_h + b_h^* = k(h,0)$,

$$(2) \qquad \begin{aligned} b_g(b_h + b_h^*) &= d(g)(b_{gh} + b_{gh}^*), \\ b_g^*(b_h + b_h^*) &= d^*(g)(b_{gh} + b_{gh}^*). \end{aligned}$$

We note that b_1 is the identity of $\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}]$.

The algebra $\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}]$ has a natural involutionary automorphism $x \mapsto \overline{x}$ induced by the mapping $g \mapsto g^{-1}$:

$$\overline{(\lambda_1 h, \lambda_2 g)} = (\lambda_1 h^{-1}, \lambda_2 g^{-1}).$$

Denote $t(g,h) := \sqrt{\frac{\varphi(h)\varphi(g)}{\varphi(gh)}}$.

PROPOSITION 2.1: *The algebra $\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}]$ is a C-algebra with respect to the distinguished basis $\mathbf{B}(G,d) = \{b_g\}_{g\in G} \cup \{b_g^*\}_{g\in G^{<k}}$. Its structure constants are given by the following formulae:*

$$
(3)\qquad
\begin{aligned}
b_g b_h &= \lambda_{g,h} b_{gh} + \lambda^*_{g,h} b^*_{gh},\\
b^*_g b_h &= (d(h) - \lambda_{g,h}) b_{gh} + (d(h) - \lambda^*_{g,h}) b^*_{gh},\\
b_g b^*_h &= (d(g) - \lambda_{g,h}) b_{gh} + (d(g) - \lambda^*_{g,h}) b^*_{gh},\\
b^*_g b^*_h &= (k + \lambda_{g,h} - d(h) - d(g)) b_{gh} + (k + \lambda^*_{g,h} - d(h) - d(g)) b^*_{gh},
\end{aligned}
$$

where

$$
(4)\qquad
\begin{aligned}
\lambda_{g,h} &= \begin{cases} \frac{d(g)d(h)+d^*(gh)t(g,h)}{k}, & \text{if } \varphi(h)\varphi(g) \neq 0,\\ \frac{d(h)d(g)}{k}, & \text{otherwise;}\end{cases}\\
\lambda^*_{g,h} &= \begin{cases} \frac{d(g)d(h)-d(gh)t(g,h)}{k}, & \text{if } \varphi(h)\varphi(g) \neq 0,\\ \frac{d(h)d(g)}{k}, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Proof: First we note that $b_1 \in \mathbf{B}$.

It is sufficient to prove the first formula from (3), since the others follow from (2).

$$b_g b_h = (d(g)g, \sqrt{\varphi(g)}g)(d(h)h, \sqrt{\varphi(h)}h) = (d(g)d(h)gh, \sqrt{\varphi(g)\varphi(h)}gh).$$

If $\varphi(g)\varphi(h) = 0$, then it follows from (1) that

$$b_g b_h = d(g)d(h)(gh, 0) = \frac{d(g)d(h)}{k}(b_{gh} + b^*_{gh})$$

and we are done.

If $\varphi(g)\varphi(h) \neq 0$, then $g, h \in G^{<k}$ which implies that $gh \in G^{<k}$. Therefore $\varphi(gh) \neq 0$ and it follows from (1) that

$$
\begin{aligned}
b_g b_h &= (d(g)d(h)gh, \sqrt{\varphi(g)\varphi(h)}gh)\\
&= d(g)d(h)\frac{b_{gh} + b^*_{gh}}{k} + \sqrt{\varphi(g)\varphi(h)}\frac{d^*(gh)b_{gh} - d(gh)b^*_{gh}}{k\sqrt{\varphi(gh)}}\\
&= d(g)d(h)\frac{b_{gh} + b^*_{gh}}{k} + t(g,h)\frac{d^*(gh)b_{gh} - d(gh)b^*_{gh}}{k}\\
&= \frac{d(g)d(h) + d^*(gh)t(g,h)}{k} b_{gh} + \frac{d(g)d(h) - d(gh)t(g,h)}{k} b^*_{gh},
\end{aligned}
$$

as desired.

If $h = g^{-1}$, then $t(g, g^{-1}) = \varphi(g)$ and

$$b_g b_{g^{-1}} = \frac{d^2(g) + (k-1)\varphi(g)}{k} b_1 + \frac{d^2(g) - \varphi(g)}{k} b^*_1.$$

Since $\varphi(g)(k-1) = d(g)d^*(g)$,

$$b_g b_{g^{-1}} = \frac{d^2(g) + d(g)d^*(g)}{k} b_1 + \frac{d^2(g) - \frac{d(g)d^*(g)}{k-1}}{k} b_1^*.$$

Now using the identity $d(g) + d^*(g) = k$ we obtain that

$$b_g b_{g^{-1}} = d(g)b_1 + \frac{d^2(g) - d(g)}{k-1} b_1^* = d(g)b_1 + (d(g) - \varphi(g))b_1^*.$$

Analogously,

$$b_g^* b_{g^{-1}}^* = d^*(g)b_1 + \frac{(d^*(g))^2 - d^*(g)}{k-1} b_1^* = d^*(g)b_1 + (d^*(g) - \varphi(g))b_1^*.$$

Thus $\lambda_{b_g \bar{b}_g b_1} = d(g)$ and $\lambda_{b_g^* \bar{b}_g^* b_1} = d^*(g)$. A routine check shows that the mapping $b_g \mapsto d(g), b_g^* \mapsto d^*(g)$ is an algebra homomorphism. ∎

PROPOSITION 2.2: *Let $(\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}], \mathbf{B}(G,d))$ be a C-algebra defined above. Then*

(1) *$(\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}], \mathbf{B}(G,d))$ is a table algebra if and only if*

$$\text{(D4)} \quad \begin{array}{l} \lambda_{g,h}^* \geq \max(0, d(g) + d(h) - k) \\ \lambda_{g,h} \leq \min(d(g), d(h)) \end{array} \iff \begin{array}{l} \frac{d(h)d(g)d^*(j)}{d^*(h)d^*(g)d(j)} \geq \frac{1}{(k-1)} \\ \frac{d^*(h)d^*(g)d^*(j)}{d(h)d(g)d(j)} \geq \frac{1}{(k-1)} \end{array}$$

holds for each triple $g, h, j \in G$ with $ghj = 1$;

(2) *if d is integral, then $(\mathbb{C}[G] \oplus \mathbb{C}[G^{<k}], \mathbf{B}(G,d))$ is a SITA if and only if it satisfies* (D4) *and the additional conditions:*

(D5) *$t(g,h) \in \mathbb{Z}$ for each pair $g, h \in G$;*

(D6) *$\frac{d(g)d(h) + d^*(gh)t(g,h)}{k} \in \mathbb{Z}$ for each pair $g, h \in G$.*

Proof: (1) A C-algebra is a table algebra iff its structure constants are non-negative. Writing the corresponding inequalities for structure constants given by (3)–(4) we obtain

$$\frac{d(g)d(h)}{d(gh)} \geq t(g,h); \quad \frac{d^*(g)d(h)}{d^*(gh)} \geq t(g,h);$$
$$\frac{d(g)d^*(h)}{d^*(gh)} \geq t(g,h); \quad \frac{d^*(g)d^*(h)}{d(gh)} \geq t(g,h).$$

Taking into account that

$$t(g,h) = \sqrt{\frac{\varphi(g)\varphi(h)}{\varphi(gh)}} = \sqrt{\frac{d(g)d^*(g)d(h)d^*(h)}{(k-1)d(gh)d^*(gh)}}$$

we obtain that being a table algebra is equivalent to the following inequalities:

$$\frac{d(g)d(h)d^*(gh)}{d^*(g)d^*(h)d(gh)} \geq \frac{1}{k-1}; \frac{d^*(g)d(h)d(gh)}{d(g)d^*(h)d^*(gh)} \geq \frac{1}{k-1};$$
$$\frac{d(g)d^*(h)d(gh)}{d^*(g)d(h)d^*(gh)} \geq \frac{1}{k-1}; \frac{d^*(g)d^*(h)d^*(gh)}{d(g)d(h)d(gh)} \geq \frac{1}{k-1}.$$

Now the statement follows from the equality $d(gh) = d((gh)^{-1})$.

(2) This part of the statement follows immediately if we impose the integrality conditions on the structure constants. ∎

2.2. Linear extensions of table subsets. Let $(A, \mathbf{B})$ be a standard table algebra and let $\mathbf{C}$ be a table subset of $\mathbf{B}$. A **linear extension** $\mathbf{L}^{\mathbf{C}}(\mathbf{B})$ of $\mathbf{C}$ is defined as a set of all $b \in \mathbf{B}$ such that $\mathsf{Supp}(b\bar{b}) \subseteq \mathbf{C}$. An equivalent definition is that $\mathbf{L}^{\mathbf{C}}(\mathbf{B})/\mathbf{C} = \mathbf{L}(\mathbf{B}/\mathbf{C})$. If $\mathbf{C} = \{1, b\}$, then we abbreviate $\mathbf{L}^b(\mathbf{B}) := \mathbf{L}^{\mathbf{C}}(\mathbf{B})$.

In this section we study linear extensions of the smallest non-trivial table subset, namely a table subset of cardinality two. Assume that $\mathbf{C} = \{1, b\}$, where b is not linear. If $\mathbf{B}$ is standard, then $|b| > 1$ and

$$b^2 = |b| \cdot 1 + (|b| - 1)b.$$

We set $k := |b| + 1$.

Lemma 2.3: *Let $(A, \mathbf{B})$ be a standard table algebra. Then the following properties hold:*

(1) *$\mathbf{L}^b(\mathbf{B})$ is a table subset of $\mathbf{B}$.*
(2) *$|x\{1, b\}| = k$ for each $x \in \mathbf{B}$ and $\mathbf{L}^b(\mathbf{B})/\{1, b\}$ is an abelian group.*
(3) *$|\mathsf{Supp}(xy)| \leq 2$ for all $x, y \in \mathbf{L}^b(\mathbf{B})$.*

Proof: Part (1) follows directly from the definition of $\mathbf{L}^b(\mathbf{B})$.

(2) Since $x\bar{x} \subset \{1, b\}$, $|x\{1, b\}| = |\{1, b\}|$ by Proposition 4.8 [AFM2]. The second part of the claim follows from the definition of $\mathbf{L}^b(\mathbf{B})$.

(3) Since $\mathbf{L}^b(\mathbf{B})/\{1, b\}$ is an abelian group, the product of two $\{1, b\}$-cosets is just one $\{1, b\}$-coset. Therefore, it is sufficient to show that each $\{1, b\}$-coset contains at most two elements.

Since $\mathbf{L}^b(\mathbf{B})/\{1, b\}$ is an abelian group, $x_i \cdot \bar{x}_j \subseteq \{1, b\}$ for each $x_i, x_j \in x\{1, b\}$. Therefore

$$x_i\bar{x}_j = \frac{|x_i|\,|x_j|}{k-1}b$$

and, consequently,

$$\langle x_i\bar{x}_i, x_j\bar{x}_j \rangle = |x_i|\,|x_j| + \frac{|x_i|\,|x_j|\,(|x_i| - 1)(|x_j| - 1)}{k-1}.$$

On the other hand,

$$\langle x_i\overline{x}_j, x_i\overline{x}_j\rangle = \frac{|x_i|^2|x_j|^2}{k-1}.$$

So

$$|x_i|\ |x_j| + \frac{|x_i|\ |x_j|(|x_i|-1)(|x_j|-1)}{k-1} = \frac{|x_i|^2|x_j|^2}{k-1}.$$

Thus

$$(k-1) + (|x_i|-1)(|x_j|-1) = |x_i|\ |x_j|$$

and

$$(k-1) - |x_i| - |x_j| + 1 = 0;$$

therefore, $|x_i|+|x_j| = k$ for each pair $i \neq j$. Now the equality $|x_1|+\cdots+|x_m| = k$ yields our statement. ∎

In what follows we'll work with a table subset $\mathbf{L}^b(\mathbf{B})$ only. For this reason we set $\mathbf{H} := \mathbf{L}^b(\mathbf{B}), A := \mathsf{Sp}(\mathbf{H})$.

The main result of this subsection follows.

THEOREM 2.4: *Let $\mathbf{B}$ be a table algebra basis, $\mathbf{C} = \{1, b\}$ a table subset with b nonlinear, $\mathbf{H} := \mathbf{L}^b(\mathbf{B})$, and $A := \mathsf{Sp}(\mathbf{H})$. Then the standard rescaling of $\mathbf{H}$ is exactly isomorphic to $\mathbf{B}(G, d)$ for a suitable finite abelian group G and degree function $d\colon G \to \mathbb{R}$ related to $|b| + 1$.*

Without loss of generality we may assume that $\mathbf{H}$ is standard (otherwise we can rescale it to standard) and $|b|$ refers to the degree of the rescaled element.

Denote $e_+ := \frac{1}{k}(1+b), e_- := 1 - \frac{1}{k}(1+b)$. It is easy to check that e_+, e_- are two pairwise orthogonal idempotents the sum of which is 1. Therefore $A = Ae_+ \oplus Ae_-$. In what follows we set $x_+ := \frac{1}{|x|}xe_+, x_- := xe_-$ for each $x \in \mathbf{H}$. Thus $\mathbf{H}_+ := \mathbf{H}/\{1, b\} = \{x_+ \colon x \in \mathbf{H}\}$. We note that the algebra Ae_+ with $\mathbf{H}_+$ as a distinguished basis is a SITA (in fact $\mathbf{H}_+$ is an abelian group).

Define a function $^*\colon \mathbf{H} \to \mathbf{H} \cup \{0\}$ by the following rule:

$$x^* := \begin{cases} y, & \text{if } x\{1,b\} = \{x, y\}; \\ 0, & \text{otherwise.} \end{cases}$$

Since each $\{1, b\}$-coset consists of at most two elements, x^* is well-defined for each $x \in \mathbf{H}$. For each $x \in \mathbf{H}$ we set $\varphi(x) := |x|(k - |x|)/(k-1)$. Since each $\{1, b\}$-coset contains at most two elements, the value of $\varphi(x)$ depends only on the $\{1, b\}$-coset of x.

PROPOSITION 2.5: *The following properties hold:*

(i) $xe_- = \frac{|x^*|x - |x|x^*}{k}$;

(ii) $\overline{x_-} = \overline{x}_-$;

(iii)

$$(x_-)(\overline{x}_-) = \varphi(x)e_-; \tag{5}$$

(iv) *if* $\mathbf{H}$ *is integral, then* $\varphi(x) = \varphi(\overline{x}) \in \mathbb{Z}$;

(v) *if* $x, y, z \in \mathbf{H}$ *are such that* $\lambda_{xyz} \neq 0$, *then* $xy = \lambda z + \lambda^* z^*$ *and*

$$\begin{aligned} x_- y_- &= (\lambda - \lambda^*) z_-; \\ \varphi(x)\varphi(y) &= (\lambda - \lambda^*)^2 \varphi(z); \end{aligned} \tag{6}$$

(vi) *If* $\mathbf{H}$ *is integral and* $x \in \mathbf{H}$ *is such that* $\gcd(|x|, |x^*|) = 1$, *then* $xx^* = \varphi(x)z$ *where* $|z| = k - 1$.

Proof: (i) Since

$$x\overline{x} = |x| \cdot 1 + \frac{|x|(|x| - 1)}{|b|} b,$$

hence $xb = (|x| - 1)x + |x|x^*$, so that

$$xe_- = x - \frac{|x|x - |x|x^*}{k} = \frac{|x^*|x - |x|x^*}{k}.$$

(ii) follows immediately from the definition of x_- and x^*.

(iii)

$$x_- \overline{x_-} = (xe_-)(\overline{x}e_-) = (x\overline{x})e_- = \left(|x| \cdot 1 + \frac{|x|(|x| - 1)}{k - 1} b\right) e_-.$$

Since $1 + b = ke_+$ and $e_+ e_- = 0$, we can rewrite the latter equality as follows:

$$\begin{aligned} x_- \overline{x_-} &= \left(|x| \cdot 1 + \frac{|x|(|x| - 1)}{k - 1}(ke_+ - 1)\right) e_- \\ &= \left(|x| - \frac{|x|(|x| - 1)}{k - 1}\right) e_- = \varphi(x) e_-. \end{aligned}$$

(iv) The equality $\varphi(x) = \varphi(\overline{x})$ follows immediately from part (iii). The inclusion $\varphi(x) \in \mathbb{Z}$ follows from the identity

$$\varphi(x) = |x| - \lambda_{x\overline{x}b}.$$

(v) Since $\mathbf{H}/\{1, b\}$ is abelian, $\mathsf{Supp}(xy) \subseteq z\{1, b\}$. If $z^* = 0$, then $z\{1, b\} = \{z\}$, and therefore $xy = \lambda_{xyz} z$. If $|\mathsf{Supp}(z\{1, b\})| = 2$, then $z\{1, b\} = \{z, z^*\}$

and $xy = \lambda z + \lambda^* z^*$ with $\lambda = \lambda_{xyz}, \lambda^* = \lambda_{xyz^*}$. In both cases we can write $xy = \lambda z + \lambda^* z^*$, where λ^* is an arbitrary integer if $z^* = 0$.

Multiplying the latter equality by e_- and taking into account that $e_-^2 = e_-$ and $(z + z^*)e_- = 0$, we obtain

$$(xe_-)(ye_-) = \lambda(ze_-) + \lambda^* z^*(e_-) = \lambda(ze_-) + \lambda^*(z + z^* - z)(e_-) = (\lambda - \lambda^*)(ze_-).$$

Applying $\overline{}$ to both sides we obtain

$$(\overline{xe_-})(\overline{ye_-}) = (\lambda - \lambda^*)(\overline{ze_-}).$$

Now multiplication of both equalities and (5) yields (6).

(vi) If x is real, then $xx^* = x\overline{x^*} = \varphi(x)b$ and we are done. So, we may assume that x is nonreal. It follows from part (iii) of Proposition 1.20 that $|\mathsf{Supp}(xx^*)| = 1$, i.e., $xx^* = \lambda z$ for some $z \in \mathbf{H}$. Therefore $\langle xx^*, xx^* \rangle = \lambda |x||x^*|$. On the other hand,

$$\langle xx^*, xx^* \rangle = \langle x\overline{x}, x^*\overline{x^*} \rangle = \langle x\overline{x^*}, x\overline{x^*} \rangle.$$

Since $x\overline{x^*} = \varphi(x)b$, $\langle x\overline{x^*}, x\overline{x^*} \rangle = \varphi(x)|x||x^*|$, and, consequently, $\lambda = \varphi(x)$, $|z| = k - 1$. ∎

In the statement below $\pi(n)$ denotes the set of prime divisors of a nonzero integer n. The set of all primes is denoted as $\mathbb{P}$.

LEMMA 2.6:

(i) *A subset* $\mathbf{X}$ *is a table subset of* $\mathbf{H}$ *if and only if either* $\mathbf{X} \le \mathbf{L}(\mathbf{B})$ *or* $\{1, b\} \subseteq \mathbf{X}$ *and there exists a subgroup* $A \le \mathbf{H}_+$ *such that* $\mathbf{X} = \{x \in \mathbf{H} \mid x_+ \in A\}$;

(ii) *if* $\mathbf{H}$ *is integral, then for each* $\Pi \subset \mathbb{P}$ *the subset* $\mathbf{H}^\Pi := \{x \mid \pi(\varphi(x)) \subseteq \Pi\}$ *is a table subset of* $\mathbf{H}$;

(iii) $\mathbf{H}^{<k} := \{x \mid |x| < k\} = \{x \mid |x\{1, b\}| = 2\} \le \mathbf{H}$;

(iv) *if* $\mathbf{H}$ *is integral and* $x_+ y_+ = z_+$ *with* $\varphi(x) = \varphi(y)$, *then* $\varphi(z)$ *is a perfect square;*

(v) *if* $\mathbf{H}$ *is integral, then* $\varphi(x)$ *is a perfect square for each* $x_+ \in (\mathbf{H}_+)^2$ *(here* A^2 *is a subgroup of* A *generated by the squares).*

Proof: (i) Since $\mathbf{X} \le \mathbf{H}$, $\mathbf{X}_+ \le \mathbf{H}_+$. If $\{1, b\} \le \mathbf{X}$, then $\mathbf{X}$ is a full preimage of $\mathbf{X}_+$ and we are done. If $\{1, b\} \not\le \mathbf{X}$, then $\mathbf{X} \cap \{1, b\} = \{1\}$ and, consequently, $\mathsf{Supp}(x\overline{x}) \subseteq \mathbf{X} \cap \{1, b\} = \{1\}$ for each $x \in \mathbf{X}$. Hence $\mathbf{X} \le \mathbf{L}(\mathbf{B})$.

(ii) Let $x, y \in \mathbf{H}$ be such that $\pi(\varphi(x)), \pi(\varphi(y)) \subseteq \Pi$. Then $\varphi(x) \neq 0, \varphi(y) \neq 0$. Write $xy = \lambda z + \lambda^* z^*$. Now the claim follows from (6).

(iii) follows from (6).

(iv) is a direct consequence from (6).

(v) is a direct consequence from (iv). ∎

It follows from part (iii) that $\mathbf{H}_+^{<k}$ is a subgroup of $\mathbf{H}_+$. A function $\rho\colon \mathbf{H}_+^{<k} \to \mathbf{H}^{<k}$ will be called **a representative function** if $\rho(x)_+ = x$ for each $x \in \mathbf{H}_+^{<k}$, equivalently, $\rho(x) \in \mathsf{Supp}(x), x \in \mathbf{H}_+^{<k}$. We always have that $\mathsf{Supp}(x) = \mathsf{Supp}(\rho(x)\{1, b\}) = \{\rho(x), \rho(x)^*\}$.

Denote

$$\rho_n(x) := \frac{1}{\sqrt{\varphi(x)}}\rho(x)_-, \quad x \in \mathbf{H}_+^{<k}.$$

Proposition 2.7: *For each* $x_1, \ldots, x_m \in \mathbf{H}_+^{<k}$,

$$\rho(x_1)\cdots\rho(x_m) = \lambda\rho(x_1\cdots x_m) + \lambda^*\rho(x_1\cdots x_m)^*,$$
$$\rho(x_1)\cdots\rho(x_m)e_- = (\lambda - \lambda^*)\rho(x_1\cdots x_m)e_-$$

and

$$\rho_n(x_1)\cdots\rho_n(x_m) = \rho_n(x_1\cdots x_m)$$

iff $\lambda - \lambda^* > 0$.

Proof: Since $\rho(x) \in \mathsf{Supp}(x)$ for each $x \in \mathbf{H}_+^{<k}$, we can write

$$\mathsf{Supp}(\rho(x_1)\cdots\rho(x_m)) \subseteq \mathsf{Supp}(x_1\cdots x_m) = \mathsf{Supp}(\rho(x_1\cdots x_m)\{1, b\}).$$

Thus $\rho(x_1)\cdots\rho(x_m) = \lambda\rho(x_1\cdots x_m) + \lambda^*\rho(x_1\cdots x_m)^*$ for suitable $\lambda, \lambda^* \in \mathbb{N}$. Multiplying both parts by e_- we obtain that

$$\rho(x_1)\cdots\rho(x_m)e_- = (\lambda - \lambda^*)\rho(x_1\cdots x_m)e_-.$$

Induction on m and (6) imply that

$$\rho_n(x_1)\cdots\rho_n(x_m) = \pm\rho_n(x_1\cdots x_m)$$

for each $m \in \mathbb{N}$. On the other hand,

$$\rho_n(x_1)\cdots\rho_n(x_m) = (\lambda - \lambda^*)\sqrt{\frac{\varphi(\rho(x_1\cdots x_m))}{\varphi(\rho(x_1))\cdots\varphi(\rho(x_m))}}\rho_n(x_1\cdots x_m).$$

Now the claim becomes evident. ∎

PROPOSITION 2.8: *There exists a representative function* $\rho\colon \mathbf{H}_{+}^{<k} \to \mathbf{H}^{<k}$ *such that* $\rho_n(h_+) = h_-$ *for each* $h \in \mathbf{L}(\mathbf{H})$ *and*

$$\rho_n(x)\rho_n(y) = \rho_n(xy) \tag{7}$$

for each $x, y \in \mathbf{H}_{+}^{<k}$.

Proof: It follows from Proposition 2.7 that

$$\rho(x)\rho(y) = \lambda\rho(xy) + \lambda^*\rho(xy)^*$$

for some $\lambda, \lambda^* \in \mathbb{N}$. It follows from Proposition 2.5 and (6) that

$$\rho(x)_-\rho(y)_- = (\lambda - \lambda^*)\rho(xy)_-$$

where

$$\lambda - \lambda^* = \pm\sqrt{\frac{\varphi(x)\varphi(y)}{\varphi(xy)}}.$$

Then

$$\rho_n(x)\rho_n(y) = \delta(x,y)\rho_n(xy) \tag{8}$$

where $\delta(x,y) \in \{\pm 1\}$. We have to show that the function ρ may be built in such a way that $\delta(x,y) = 1$ for each $x, y \in \mathbf{H}_{+}^{<k}$ and $\rho_n(h_+) = h$ for every $h \in \mathbf{L}(\mathbf{H})$.

Since $\mathbf{L}(\mathbf{H})_+$ is a subgroup of $\mathbf{H}_{+}^{<k}$, there exists a decomposition $\mathbf{H}_{+}^{<k} = X_1 \times \cdots \times X_r$ into the product of cyclic subgroups such that

$$\mathbf{L}(\mathbf{H})_+ = (\mathbf{L}(\mathbf{H})_+ \cap X_1) \times \cdots \times (\mathbf{L}(\mathbf{H})_+ \cap X_r).$$

Let x_i and y_i be generators of X_i and $\mathbf{L}(\mathbf{H}) \cap X_i$, respectively. Denote by n_i the order of x_i in the factor-group $X_i/(\mathbf{L}(\mathbf{H})_+ \cap X_i)$. It follows from (8) that $\rho_n(x_i)^{n_i} = \pm\rho_n(x_i^{n_i}) = \pm g_i e_-$, where g_i is a unique linear element of $\mathsf{Supp}(x_i^{n_i})$. It is clear that $x_i^{n_i} = y_i = (g_i)_+$.

Define now

$$\sigma(x_i) := \begin{cases} \rho(x_i) & \text{if } \rho_n(x_i)^{n_i} = +g_i e_-, \\ \rho(x_i)^* & \text{if } \rho_n(x_i)^{n_i} = -g_i e_-. \end{cases}$$

If n_i is even, then

$$(\rho(x_i^{n_i/2}))^2 = \lambda g_i + \lambda^* g_i^*.$$

Since g_i is linear, $\lambda = |\rho(x_i^{\frac{n_i}{2}})| > \lambda^*$.‡

Define $\rho(h_+) = h$ for all $h \in \mathbf{L}(\mathbf{H})$. Then $\rho_n(h_+) = he_-$, and hence $(\rho_n(x_i^{n_i/2}))^2 = \rho_n(x_i^{n_i}) = g_i e_-$.

Now the equality

$$\rho_n(x_i)^{n_i/2} = \pm\rho_n(x_i^{n_i/2})$$

implies that

$$\rho_n(x_i)^{n_i} = (\rho_n(x_i)^{n_i/2})^2 = (\pm\rho_n(x_i^{n_i/2}))^2 e_- = g_i e_-.$$

Thus if $\rho_n(x_i)^{n_i} = -g_i e_-$, then n_i is odd. Therefore $\sigma_n(x_i) = -\rho_n(x_i)$, and, consequently, $\sigma_n(x_i)^{n_i} = g_i e_-$.

Thus $\sigma_n(x_i)^{n_i} = g_i e_-$ for each $i = 1, \ldots, r$. Denote $N_i := o(x_i)$. Clearly $N_i = n_i o(g_i)$. It follows from Proposition 2.7 that for each $n \in [0, N_i - 1]$ we have $\sigma(x_i)^n = \lambda h + \lambda^* h^*$ for suitable $h \in \mathbf{H}$. Set

$$\sigma(x_i^n) := \begin{cases} h & \text{if } \lambda^* \leq \lambda, \\ h^* & \text{otherwise.} \end{cases} \tag{9}$$

By Proposition 2.7, $\sigma_n(x_i^n) := (\sigma_n(x_i))^n$ for each $n \in [0, N_i - 1]$. Since $x_i^{N_i} = 1$ and $(\sigma_n(x_i))^{N_i} = (g_i)^{o(g_i)} e_- = e_-$, the equality $\sigma_n(x_i^n) := (\sigma_n(x_i))^n$ holds for each integer n.

If $h \in \mathbf{L}(\mathbf{H})$ is such that $h_+ \in \mathbf{L}(\mathbf{H})_+ \cap X_i$, then $h = g_i^m$ for some m and

$$\begin{aligned} \sigma_n(h_+) &= \sigma_n((g_i^m)_+) = \sigma_n(((g_i)_+)^m) \\ &= \sigma_n(x_i^{n_i m}) = \sigma_n(x_i)^{n_i m} = (g_i e_-)^m = he_-. \end{aligned}$$

Therefore $\sigma(h_+) = h$.

If $(x_1)^{m_1} \cdot \ldots \cdot (x_r)^{m_r} \in \mathbf{H}_+^{<k}$ is an abritrary element, then

$$\sigma(x_1^{m_1}) \cdot \ldots \cdot \sigma(x_r^{m_r}) = \lambda h + \lambda^* h^*$$

for a suitable $h \in \mathbf{H}_+^{<k}$. As before, we set

$$\sigma(x_1^{m_1} \cdots x_r^{m_r}) := \begin{cases} h & \text{if } \lambda^* \leq \lambda, \\ h^* & \text{otherwise.} \end{cases} \tag{10}$$

‡ Here we used the fact that if $u \in \mathbf{B}$ is linear and $\lambda_{vwu} \neq 0$, then $|v| = |w| = \lambda_{vwu}$. We didn't find the proof of this claim in the literature, so we reproduce here the proof proposed by the referee.

By Proposition 1.20, $\lambda_{vwu}|u| = \lambda_{u\overline{w}v}|v| \iff \lambda_{vwu} = \lambda_{\overline{w}uv}|v|$. By [AB, Proposition 3.2] $|\mathsf{Supp}(u\overline{w})| = 1$. Therefore $u\overline{w} = \lambda_{u\overline{w}v} v$. It follows from $\lambda_{u\overline{w}v}|v| = |u\overline{w}| = |\overline{w}| = \langle w\overline{w}, u\overline{u}\rangle = \langle u\overline{w}, u\overline{w}\rangle = \lambda_{u\overline{w}v}^2|v|$ that $\lambda_{u\overline{w}v} = 1$. Therefore $\lambda_{vwu} = |v|$.

Thus

$$\sigma_n(x_1^{m_1})\cdot\ldots\cdot\sigma_n(x_r^{m_r})=\sigma_n(x_1^{m_1}\cdots x_r^{m_r}).$$

Now we can write

$$\begin{aligned}\sigma_n(x_1^{p_1}\cdots x_r^{p_r})\sigma_n(x_1^{q_1}\cdots x_r^{q_r})&=\sigma_n(x_1^{p_1})\cdot\ldots\cdot\sigma_n(x_r^{p_r})\sigma_n(x_1^{q_1})\cdot\ldots\cdot\sigma_n(x_r^{q_r})\\&=\sigma_n(x_1)^{p_1}\cdot\ldots\cdot\sigma_n(x_r)^{p_r}\sigma_n(x_1)^{q_1}\cdot\ldots\cdot\sigma_n(x_r)^{q_r}\\&=\sigma_n(x_1)^{p_1+q_1}\cdot\ldots\cdot\sigma_n(x_r)^{p_r+q_r}\\&=\sigma_n(x_1^{p_1+q_1})\cdot\ldots\cdot\sigma_n(x_r^{p_r+q_r})\\&=\sigma_n(x_1^{p_1+q_1}\cdots x_r^{p_r+q_r}).\end{aligned}$$

Consider an arbitrary element $h\in\mathbf{L}(\mathbf{H})$. Then $h=y_1^{p_1}\cdot\ldots\cdot y_r^{p_r}$ for some p_i's. Then

$$\begin{aligned}\sigma_n(h_+)&=\sigma_n((g_1)_+^{p_1}\cdot\ldots\cdot(g_r)_+^{p_r})=\sigma_n((g_1)_+^{p_1})\cdot\ldots\cdot\sigma_n((g_r)_+^{p_r})\\&=(g_1)^{p_1}e_-\cdot\ldots\cdot(g_r)^{p_r}e_-=y_1^{p_1}\cdot\ldots\cdot y_r^{p_r}e_-=he_-.\end{aligned}$$

Therefore $\sigma(h_+)=h$. ∎

PROPOSITION 2.9: *Let ρ: $\mathbf{H}_+^{<k}\to\mathbf{H}^{<k}$ be a representative function which satisfies (7). Then $\mathbf{H}\simeq_x\mathbf{B}(\mathbf{H}_+,d)$, where d is a degree function related to k defined as follows:*

for all $h\in\mathbf{H}^{<k}$,

$$d(h)=\begin{cases}|\rho(h_+)| & \text{if } h=\rho(h_+),\\ k-|\rho(h_+)| & \text{if } h=\rho(h_+)^*.\end{cases}\tag{11}$$

Proof: First we check that d satisfies the conditions (D1)–(D3).

If $x\in\mathbf{H}_+^{<k}$, then $|\rho(x)|<k$ follows from the definition of $\mathbf{H}^{<k}$. If $|\rho(x)|=k-1$, then $|\rho(x)^*|=1$ and, by Proposition 2.8, $\rho(\rho(x)_+^*)=\rho(x)^*$, contrary to $\rho(\rho(x)_+^*)=\rho(\rho(x)_+)=\rho(x)$. Therefore $|\rho(x)|\in[1,k-1)$. Since $|\rho(1)|\in\{1,k-1\}$, $|\rho(1)|=1$ implies that $\rho(1)=1$. By Proposition 2.8, $\rho_n(x)\rho_n(x^{-1})=\rho_n(1)=e_-$. Therefore $\rho(x^{-1})\in\{\overline{\rho}(x),\overline{\rho}(x)^*\}$. If $\rho(x^{-1})=\overline{\rho}(x)^*$, then $\rho_n(x)\rho_n(x^{-1})=-e_-$, which is impossible. Thus $\rho(x^{-1})=\overline{\rho}(x)$ and, therefore, $|\rho(x)|=|\rho(x^{-1})|$. Thus (D1)–(D3) are satisfied.

Define a linear mapping

$$f\colon\mathbb{C}[\mathbf{H}_+]\oplus\mathbb{C}[\mathbf{H}_+^{<k}]\to A$$

via its action on the basis $\{(h,0)\}_{h\in\mathbf{H}_+}\cup\{(0,h)\}_{h\in\mathbf{H}_+^{<k}}$:

$$f((h,0))=h,f((0,h))=\rho_n(h).$$

Since $A_+ = Ae_+$ is a table algebra with a distinguished basis $\mathbf{H}_+$, f is an isomorphism between $(\mathbb{C}[\mathbf{H}_+], 0)$ and A_+.

Since ρ_n satisfies (7), ρ_n is a group homomorphism from $\mathbf{H}_+^{<k}$ into A_-. Its kernel is trivial, since the elements $\rho_n(h), h \in \mathbf{H}_+^{<k}$ are linearly independent. Thus $\{\rho_n(h)\}_{h\in\mathbf{H}_+^{<k}}$ is a group isomorphic to $\mathbf{H}_+^{<k}$. Thus f is an isomorphism between $(0, \mathbb{C}[\mathbf{H}_+^{<k}])$ and A_-. Combined altogether we obtain that f is an algebra isomorphism between $\mathbb{C}[\mathbf{H}_+] \oplus \mathbb{C}[\mathbf{H}_+^{<k}]$ and A. To finish the proof we need to check that it maps the distinguished basis of $\mathbb{C}[\mathbf{H}_+] \oplus \mathbb{C}[\mathbf{H}_+^{<k}]$ onto $\mathbf{H}$. If $\varphi(h) = 0$, then

$$f(b_h) = f(d(h)h, 0) = d(h)h = kh \in \mathbf{H}.$$

If $\varphi(h) \neq 0$, then

$$f(b_h) = f(d(h)h, \sqrt{\varphi(h)}h) = d(h)h + \sqrt{\varphi(h)}\rho_n(h) = d(h)h + \rho(h)_-.$$

Since $h = \rho(h)_+$ and $d(h) = |\rho(h)|$,

$$f(b_h) = |\rho(h)|\rho(h)_+ + \rho(h)_- = \rho(h).$$

Analogously,

$$\begin{aligned} f(b_h^*) &= f(d^*(h)h, -\sqrt{\varphi(h)}h) = d^*(h)h - \sqrt{\varphi(h)}\rho_n(h) \\ &= d^*(h)h - \rho(h)_- = |\rho(h)^*|\rho(h)_+ - \rho(h)_- = \rho(h)^*. \end{aligned}$$ ∎

As a corollary we obtain the following statement:

PROPOSITION 2.10: *Let $a \in \mathbf{B}$ be an element of width 2 such that $a\bar{a} = k \cdot 1 + kb$, $|a| = k, |b| = k - 1$. Then $\mathbf{B}_a \simeq_x \mathbf{B}(\mathbb{Z}_m, d)$ for some $m \in \mathbb{N}$ and degree function $d: \mathbb{Z}_m \to \mathbb{R}$.*

Proof: $\langle a, ab\rangle = \langle a\bar{a}, b\rangle = k(k-1)$ implies that $ab = (k-1)a$, and hence $b \in \operatorname{sta} a$. Thus, $\operatorname{sta} a = \{1, b\}$, so this is table subset. Now Theorem 2.4 applies. ∎

3. Enumeration of degree functions related to a given k

In general it is difficult to describe all degree functions related to $k \in \mathbb{N}$. The first natural question which naturally arises is: what can be said about the cardinality of $\mathsf{Im}(d)$? The following statement is easy to check.

PROPOSITION 3.1: *Let G be an abelian group and d be a degree function such that $\mathsf{Im}(d) = \{1\}$. Then d satisfies the conditions (D1)–(D6).*

Since $d(g)$ is an idempotent in $\mathbb{Z}_{k-1}$ we get the following

PROPOSITION 3.2: $|\mathsf{Im}(d)| \leq 2^{|\pi(k-1)|}$.

Let $x \in \mathbf{H}_+$ be an arbitrary element. Set

$$a := d(x),\ b := d(x^2),\ a^* := d^*(x),\ b^* := d^*(x^2),\ \lambda := \lambda_{x,x},\ \lambda^* := \lambda^*_{x,x}.$$

Then

$$a^2 = \lambda b + \lambda^* b^*; \quad aa^* = \alpha b + \alpha^* b^*, \tag{12}$$

where $\alpha := a - \lambda, \alpha^* := a - \lambda^*$. Since $\lambda > \lambda^*$, $\alpha < \alpha^*$.

Then the conditions (D1)–(D6) imply the following equations, which will be referred to as **E-equations**:

$$\text{(13)}\qquad \begin{array}{ll} \text{(a)} & a + a^* = b + b^* = k, \\ \text{(b)} & a \geq \lambda, \\ \text{(c)} & \max(0, 2a - k) \leq \lambda^*, \\ \text{(d)} & \varphi(a) := aa^*/(k-1) \in \mathbb{N}, \\ \text{(e)} & \varphi(b) := bb^*/(k-1) \in \mathbb{N}, \\ \text{(f)} & t := \varphi(a)/\sqrt{\varphi(b)} \in \mathbb{N}, \\ \text{(g)} & \lambda = (a^2 + b^* t)/k \in \mathbb{N}, \\ \text{(h)} & \gcd(a, a^*) \neq 1. \end{array}$$

A solution with $a = 1$ or $b = 1$ is called trivial. By part (vi) of Proposition 2.5 nontriviality of a solution implies that $\gcd(a, a^*) \neq 1$, which explains part (h) of (13).

If $a = b$ is a solution of E-equations, then we have the following

THEOREM 3.3: *Let $F \leq H \leq G$ be abelian groups. Let $a = b \in \mathbb{N}$ be a non-trivial solution of (13). Then the function d: $H \to \mathbb{N}$ defined by*

$$d(h) = \begin{cases} a, & \text{if } h \in H \setminus F \\ 1, & \text{if } h \in F \end{cases}$$

is a degree function related to k which satisfies the conditions (D1)–(D6).

Proof: The conditions (D1)–(D3) follow immediately.

To check (D4) pick an arbitrary triple $i, j, k \in G$ which satisfies $ijk = 1$. Then $(d(i), d(j), d(k))$ is one of the following triples: $(1,1,1), (a,a,1)$, (a,a,a). In the first two cases (D4) is evident. In the remaining case (D4) follows from (13)(b).

(D5) Computing $t(g,h)$ we obtain

$$t(g,h) = \begin{cases} 1, & g,h \in F; \\ 1, & g \in F, h \in H \setminus F; \\ \frac{a(k-a)}{k-1}, & g,h \in H \setminus F, gh \in F; \\ \sqrt{\frac{a(k-a)}{k-1}}, & g,h \in H \setminus F, gh \in H \setminus F. \end{cases}$$

It follows from (13)(f),(d) that $t(g,h) \in \mathbb{N}$ for each $g,h \in H$. Thus (D5) is satisfied.

(D6) The number $\frac{d(g)d(h)+d^*(gh)t(g,h)}{k}$ is 1 if one of g,h does not belong to F. If $g,h \in F$, then $\frac{d(g)d(h)+d^*(gh)t(g,h)}{k} \in \mathbb{N}$ by Part (g) of (13). ∎

Not every solution of E-equations has a property $a = b$. The statement below yields an infinite series of solutions with $a \neq b$.

THEOREM 3.4: *Let $1 \leq H_1 \leq H_2 \leq H_3 \leq G$ be abelian groups. For each $m \in \mathbb{N}$ set*

$$q := 4m^2 - 2, \quad s := (q^2-1)m, \quad k := 4s^2 = (q^2-1)^2 4m^2,$$
$$a := (q^2-1)(q^2+q-1), \quad b := s(2s+\delta),$$

where δ is -1 if s is odd and 1 otherwise. Then d: $G \to \mathbb{N}$ defined below is a degree function related to k.

$$d(g) := \begin{cases} 1, & g \in H_1, \\ a, & g \in H_2 \setminus H_1, \\ b, & g \in H_3 \setminus H_2. \end{cases}$$

Proof: The conditions (D1)–(D3) are evident. The conditions (D4)–(D6) are equivalent to the fact that $\lambda_{x,y}, \lambda^*_{x,y}, x,y \in G$ are non-negative integers which satisfy

$$\lambda_{x,y} \leq d(x), d(y); \quad \lambda^*_{x,y} \geq 0, \ d(x)+d(y)-k. \tag{14}$$

Direct calculations show that $\lambda_{x,y}, \lambda^*_{x,y}$ have the values given in Table 1 (recall that $\lambda^*_{x,y} = \lambda_{x,y} - t(x,y)$).

Table 1

$d(x)$	$d(y)$	$d(xy)$	$t(x,y)$	$\lambda_{x,y}$
1	1	1	1	1
a	a	1	q^2-1	a
b	b	1	s	b
a	1	a	1	1
b	1	b	1	1
a	a	a	q^2-1	q^3+q^2-2q
b	b	b	s	$s^2+\delta s+\frac{s+1-\delta}{2}$
b	a	b	q^2-1	$(q-1)(q+1)^2(2m^2+1)-m$
b	b	a	$(q^2-1)m^2$	$s^2+s\delta+q(q-1)m^2$

Now one can check that $\lambda_{x,y}, \lambda^*_{x,y}$ satisfy (14). ∎

In the next subsection we will find all solutions of (13) with $a = b$.

3.1. Solutions of the E-equations. Set

$$l := \gcd(a, k-1), \quad l^* := \gcd(a^*, k-1), \quad m := \frac{a}{l}, \quad m^* := \frac{a^*}{l^*}.$$

Clearly, $a = ml, a^* = m^*l^*$.

Proposition 3.5: *The following properties hold:*

(1) $\gcd(m, l^*) = \gcd(m^*, l) = \gcd(l^*, l) = 1$;

(2) $ll^* = k-1$ *and, therefore,* $ml + m^*l^* = ll^* + 1$;

(3) $d := \gcd(m, k) = \gcd(m^*, k) = \gcd(m, m^*)$.

Proof: (1)

$$k = ml + m^*l^* \Longrightarrow ml \equiv 1 (\operatorname{mod} l^*) \Rightarrow \gcd(m, l^*) = 1, \gcd(l, l^*) = 1.$$

$$k = ml + m^*l^* \Longrightarrow m^*l^* \equiv 1 (\operatorname{mod} l) \Rightarrow \gcd(m^*, l) = 1, \gcd(l, l^*) = 1.$$

(2) Since l and l^* are coprime divisors of $k-1$, $l^* \mid (k-1)/l$. Since $a^2 - a$ is divisible by $k-1$ (see (d) of (13)), $ml(ml-1) \equiv 0(\operatorname{mod} k-1) \Longrightarrow m(ml-1) \equiv 0(\operatorname{mod} \frac{k-1}{l})$. Now the equality $\gcd(a, k-1) = l$ implies that $\gcd(m, \frac{k-1}{l}) = 1$. Therefore

$$(ml-1) \equiv 0\Big(\operatorname{mod} \frac{k-1}{l}\Big) \Longrightarrow a^* \equiv 0\Big(\operatorname{mod} \frac{k-1}{l}\Big)$$
$$\Longrightarrow \frac{k-1}{l} \mid a^* \Longrightarrow \frac{k-1}{l} \mid l^*.$$

(3) It follows from (1) that $\gcd(a,a^*) = \gcd(m,m^*)$.

It follows from $l \mid (k-1)$ that $\gcd(a,k) = \gcd(ml,k) = \gcd(m,k)$. Analogously, $\gcd(a^*,k) = \gcd(m^*l^*,k) = \gcd(m^*,k)$. Now the claim follows from $\gcd(a,k) = \gcd(a^*,k) = \gcd(a,a^*)$. ∎

It follows from part (f) of the E-equations that $(\alpha - \alpha^*)^2 = t^2 = \varphi(a)$. Therefore

$$(\lambda - \lambda^*)^2 = (\alpha - \alpha^*)^2 = \varphi(a) = \frac{aa^*}{k-1} = mm^*.$$

Thus mm^* is a perfect square. Since m/d and m^*/d are coprime, $m/d = u^2, m^*/d = (u^*)^2$ for some coprime numbers u, u^*. Thus we obtain that

$$\frac{k}{d} = l^*{u^*}^2 + lu^2$$

and all the numbers $l, l^*, u, u^*, k/d$ are pairwise coprime.

It follows from $aa^* = \alpha a + \alpha^* a^*$ (see (12)) that a^*/d divides α and a/d divides α^*. Write $\alpha = \mu(a^*/d) = \mu l^*(u^*)^2, \alpha^* = \mu^*(a/d) = \mu^* lu^2$. By the E-equations

$$\alpha - \alpha^* = -t \Rightarrow \mu l^*(u^*)^2 - \mu^* lu^2 = -duu^*$$

and

$$\mu + \mu^* = d.$$

$\mu(u^*)^2 - \mu^* u^2 = duu^*$. Thus $\mu = \nu u, \mu^* = \nu^* u^*$ and, therefore,

$$\begin{cases} \nu u + \nu^* u^* = d; \\ \nu l^* u^* - \nu^* lu = -d. \end{cases}$$

Finally we obtain that

$$\nu = d\frac{lu - u^*}{lu^2 + l^*(u^*)^2}; \quad \nu^* = d\frac{l^*u^* + u}{lu^2 + l^*(u^*)^2}$$

which implies that

$$\frac{lu^2 + l^*(u^*)^2}{d} = \frac{l^*u^* + u}{\nu^*} = \frac{lu - u^*}{\nu}.$$

Set

$$w := \frac{lu^2 + l^*(u^*)^2}{d}$$

(w may be not an integer). This gives us the following:

$$l = \frac{\nu w + u^*}{u}; \quad l^* = \frac{\nu^* w - u}{u^*}.$$

Taking into account that $ll^* + 1 = k = d(u^2 l + (u^*)^2 l^*) = d^2 w$ we obtain

$$w = \frac{d^2 uu^* - \nu^* u^* + \nu u}{\nu\nu^*},$$

where $d = \nu u + \nu^* u^*$. Therefore

$$l = \frac{d^2 u^* + \nu}{\nu^*}; \quad l^* = \frac{d^2 u - \nu^*}{\nu}.$$

THEOREM 3.6: *If $u, u^*, \nu, \nu^* \in \mathbb{N}$ are such that $\gcd(u, u^*) = 1$ and*

$$\begin{cases} \nu^2 u^2 u^* \equiv -\nu (\operatorname{mod} \nu^*), \\ (\nu^*)^2 u (u^*)^2 \equiv \nu^* (\operatorname{mod} \nu), \end{cases} \tag{15}$$

then the following numbers yield a solution of the E-equations:

$$a = b = du^2 l, \quad a^* = b^* = d(u^*)^2 l^*, \quad \alpha = \nu l^* u (u^*)^2, \quad \alpha^* = \nu^* l u^* u^2,$$

where $d = u\nu + u^*\nu^*$ and $l = \frac{d^2 u^* + \nu}{\nu^*}$; $l^* = \frac{d^2 u - \nu^*}{\nu}$.

Proof: First we note that $l, l^* \in \mathbb{N}$ by (15).

Further

$$\begin{aligned} k = a + a^* &= d(u^2 l + (u^*)^2 l^*) \\ &= d\Big(u^2 \frac{d^2 u^* + \nu}{\nu^*} + (u^*)^2 \frac{d^2 u - \nu^*}{\nu}\Big) \\ &= d\frac{d^2 u^2 u^* \nu + (\nu u)^2 + d^2 u (u^*)^2 \nu^* - (\nu^* u^*)^2}{\nu\nu^*} \\ &= d\frac{d^2 uu^*(u\nu + u^*\nu^*) + (\nu^* u^* + \nu u)(-\nu^* u^* + \nu u)}{\nu\nu^*} \\ &= d\frac{d^3 uu^* + d(\nu^* u^* - \nu u)}{\nu\nu^*} = ll^* + 1. \end{aligned}$$

(b) is equivalent to $\alpha \geq 0$, which holds trivially.

(c) is equivalent to $\alpha^* \leq \min(a, a^*)$. The inequality $\alpha^* \leq a$ is evident. So we need to check only that $\alpha^* \leq a^*$. Now we can write

$$\begin{aligned} &\alpha^* \leq a^* \iff \nu^* l u^* u^2 \leq d(u^*)^2 l^* \iff (d^2 u^* + \nu) u^2 \leq du^* \frac{d^2 u - \nu^*}{\nu} \\ &\iff (d^2 uu^* + \nu u)\nu u \leq d^3 uu^* - d\nu^* u^* \iff (\nu u)^2 + d\nu^* u^* \leq d^2 uu^* \nu^* u^*. \end{aligned}$$

Now the claim follows from

$$(\nu u)^2 + d\nu^* u^* = (\nu u)^2 + (\nu u + \nu^* u^*)\nu^* u^* \leq (\nu u + \nu^* u^*)^2 = d^2 \leq d^2 uu^* \nu^* u^*.$$

(d)+(e)+(f) We have

$$\varphi(a) = \varphi(b) = (duu^*)^2 \Rightarrow t = duu^*.$$

(g) It is enough to show that $a - \lambda \in \mathbb{Z}$,

$$\begin{aligned} a - \lambda = a - \frac{a^2 + a^*t}{k} &= \frac{a^*(a-t)}{k} \\ &= \frac{d(u^*)^2l^*(du^2l - duu^*)}{k} = \frac{u(u^*)^2l^*(ul - u^*)}{k/d^2}. \end{aligned}$$

Since

$$ul - u^* = \frac{d^2uu^* + u\nu - u^*\nu^*}{\nu^*} \quad \text{and} \quad k/d^2 = \frac{d^2uu^* + u\nu - u^*\nu^*}{\nu\nu^*}$$

we obtain $a - \lambda = \nu u(u^*)^2l^* \in \mathbb{Z}$.

(h) Since ν, ν^*, u, u^* are natural numbers, $d \geq 2$, which implies that $\gcd(a, a^*) \geq 2$.

It is not difficult to describe all solutions of (15). Set $n := \gcd(\nu, \nu^*)$, $\nu = n\theta$, $\nu^* = n\theta^*$. Then $\gcd(\theta, \theta^*) = 1$ and (15) are equivalent to the following congruences:

$$\text{(16)} \qquad \begin{cases} n\theta u^2u^* \equiv -1 (\operatorname{mod} \theta^*); \\ n\theta^*u(u^*)^2 \equiv 1 (\operatorname{mod} \theta). \end{cases}$$

Thus we have a quadruple of pairwise coprime numbers u, u^*, θ, θ^*. By the Chinese remainder theorem the congruences (16) always have an infinite series of solutions for each quadruple u, u^*, θ, θ^* of pairwise coprime natural numbers.

Some infinite series of solutions of (15) may be easily pointed out:

$\theta = 1, \theta^* = 1$ and $n, u, u^* \in \mathbb{N}$ are arbitrary numbers with $\gcd(u, u^*) = 1$;

$u = u^* = 1, n = 1, \theta^* = \theta + 1, \theta \in \mathbb{N}$.

Table 2 contains the list of solutions to the E-equations for $a = b, a^* = b^*$ as solved by Shlomo Arad's computer program for values of $1 \leq k \leq 2000$. One can see that for each $k \leq 2000$ there exist at most one solution. Some of the solutions found belong to the infinite series defined above.

Table 2

u	u^*	θ	θ^*	n	α	α^*	a	a^*	k
1	1	1	1	1	3	5	10	6	16
1	1	1	2	1	7	10	15	21	36
1	1	2	1	1	8	11	33	12	45
1	1	1	1	2	14	18	36	28	64
1	1	3	1	1	15	19	76	20	96
1	1	2	3	1	22	27	45	55	100
1	1	1	1	3	33	39	78	66	144
1	2	1	1	1	32	38	57	96	153
2	1	1	1	1	34	40	120	51	171
1	1	4	1	1	24	29	145	30	175
1	1	3	4	1	45	52	91	105	196
1	1	1	1	4	60	68	136	120	256
1	1	5	1	1	35	41	246	42	288
1	1	1	3	2	58	66	88	232	320
1	1	4	5	1	76	85	153	171	324
1	1	1	2	3	75	84	126	225	351
1	1	2	1	3	78	87	261	117	378
1	1	1	1	5	95	105	210	190	400
1	2	1	1	1	96	106	265	160	425
1	1	6	1	1	48	55	385	56	441
1	1	5	6	1	115	126	231	253	484
1	1	1	1	6	138	150	300	276	576
1	2	1	1	2	136	148	222	408	630
1	1	7	1	1	63	71	568	72	640
2	1	1	1	2	140	152	456	210	666
1	1	6	7	1	162	175	325	351	676
1	3	1	1	3	135	147	540	196	736
1	1	1	1	7	189	203	406	378	784
3	1	1	1	1	141	153	612	188	800
1	1	8	1	1	80	89	801	90	891
1	1	7	8	1	217	232	435	465	900
1	3	2	1	1	216	231	385	540	925
2	1	1	5	1	186	200	280	651	931
3	1	1	2	1	219	234	585	365	950
1	1	1	2	5	215	230	345	645	990
1	1	1	1	8	248	264	528	496	1024
1	1	2	1	5	220	235	705	330	1035
1	1	8	9	1	280	297	561	595	1156
1	1	9	1	1	99	109	1090	110	1200
1	1	1	1	9	315	333	666	630	1296
1	1	1	4	3	213	228	285	1065	1350
1	1	3	1	4	252	268	1072	336	1408
1	2	1	1	3	312	330	495	936	1431
1	1	9	10	1	351	370	703	741	1444
2	1	1	1	3	318	336	1008	477	1485
1	2	1	3	1	184	198	231	1288	1519
1	1	10	1	1	120	131	1441	132	1573
1	1	1	1	10	390	410	820	780	1600
2	1	1	3	2	388	408	680	970	1650
1	1	10	11	1	430	451	861	903	1764
1	3	4	1	1	432	453	1057	756	1813
1	1	1	1	11	473	495	990	946	1936
1	1	1	2	7	427	448	672	1281	1953

PROOF OF THE MAIN THEOREMS.

3.2. PROOF OF THEOREM 1.1. Recall that $a \in \mathbf{B}^{\#}$ is an element of width 2 and $|a| = k$.

(I) If $k = 2$, then $\mathbf{B}$ contains elements of degrees 1, 2 and $\mathbf{B}$ can be rescaled to a HITA of degree 2, and by Lemma 1.21 Case I holds.

(II) If $k \geq 3$, let $a\overline{a} = k \cdot 1 + \lambda b$, $b \in \mathbf{B}^{\#}$; then $\lambda|b| = k^2 - k$, $\langle a\overline{a}, b\rangle = \lambda|b| = k(k-1)$, $\langle a, ab\rangle = \lambda_{aba} \cdot k$. So $ab = (k-1)a + \sum_{x \neq a} \lambda_{abx} x$. Since $|b| \leq k$, we have $\sum_{x\neq a} \lambda_{abx}|x| \leq k$. On the other hand, $k \mid \lambda_{abx}|x|$.

Now for all $x \in \mathsf{Supp}(ab) \setminus \{a\}$, we have one of the following cases:

CASE 1: $\lambda_{abx}|x| = k$, $|b| = k$, $\lambda = k-1$.

CASE 2: $\lambda_{abx}|x| = 0$, $|b| = k-1$, $\lambda = k$.

CASE 1: Let $b^2 = k \cdot 1 + \beta b + u$ and $ab = (k-1)a + \alpha x$ where $\beta \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{N}$, $u \in A$. Thus $\alpha|x| = |a|\,|b| - (k-1)|a| = k$. Now $\langle a\overline{a}, b^2\rangle = k^2 + \beta\lambda|b| = k^2 + \beta k(k-1)$ and $\langle ab, ab\rangle = (k-1)^2|a| + \alpha^2|x| = (k-1)^2 k + \alpha k$. Therefore $k + \beta(k-1) = (k-1)^2 + \alpha$. Then $1 \leq \alpha \leq k$ implies that $k-2 \leq \beta \leq k-1$, hence $\beta \in \mathbb{N}$.

SUBCASE (a): $\beta = k-1$. Thus $\alpha = k$ and we have that $b^2 = k \cdot 1 + (k-1)b$. Thus $xb = a$ and $x \in \mathbf{B}$ is linear. Since $B_x \cdot B_b = B_a$ and $B_x \cap B_b = \{1\}$, we have (Lemma 1.16) that $\mathbf{B}_a \simeq_x (Z_m \otimes \mathbf{T}_0(k))'$ for some $m \in \mathbb{N}$, where

$$T_0(k) = \{1, b\},\ b^2 = k \cdot 1 + (k-1)b,$$

and case II(a) holds.

SUBCASE (b): $\beta = k-2$, $\alpha = 1$, $|x| = k$, and we have $b^2 = k \cdot 1 + (k-2)b + u$, $u \in A$, $|u| = k$, $ab = (k-1)a + x$.

$|u| = |b|$ and $|b| \mid \gamma|c|$ for any $c \in \mathbf{B}$ that appears in u with coefficient γ. Then $u = \gamma c$ for some $c \in \mathbf{B}^{\#}$, $\gamma \in \mathbb{N}$ and $u = \overline{u}$. Now

$$(a\overline{a})b = kb + (k-1)b^2 = kb + (k-1)[k + (k-2)b + u],\ (ab)\overline{a} = (k-1)a\overline{a} + \overline{a}x,$$

but $\lambda_{\overline{a}xc} \leq |a| = k$ by [AFM2, Proposition 2.3]. Thus $\overline{a}x = b + (k-1)u$ and $u \in \mathbf{B}^{\#}$.

$$(\overline{a}x)b = x(\overline{ab}) = x((k-1)\overline{a} + \overline{x}) = (k-1)b + (k-1)^2 u + x\overline{x},$$
$$(\overline{a}x)b = (b + (k-1)u)b = b^2 + (k-1)bu = k \cdot 1 + (k-2)b + u + (k-1)bu.$$

Therefore $bu = b + (k-1)u$, $u^2 = u\overline{u} = k \cdot 1 + (k-1)b$, $au = (k-1)x + \delta z$ for some $\delta \in \mathbb{N}$ and $z \in \mathbf{B}^{\#}$ where $\delta|z| = k$.

Now $\langle u\bar{u}, a\bar{a}\rangle = \langle au, au\rangle$ holds.

$$\langle u\bar{u}, a\bar{a}\rangle = k^2 + (k-1)^2k, \quad \langle au, au\rangle = (k-1)^2k + \delta^2|z|.$$

Thus $k^2 = \delta^2|z| = \delta k$; then $\delta = k$ and $|z| = 1$, so $z\bar{u} = a$. Thus $B_z \cdot B_u = B_a$, $B_z \cap B_u = \{1\}$, therefore $B_a \simeq Z_m \otimes V(3,k)$ for some $m \in \mathbb{N}$, $V(3,k) = \{1, b, u\}$,

$$b^2 = k\cdot 1 + (k-2)b + u,$$
$$u^2 = k\cdot 1 + (k-1)b,$$
$$bu = b + (k-1)u,$$

and case II(b) holds.

CASE 2: $|b| = k-1$, $ba = (k-1)a$. Thus $a\bar{a} = k\cdot 1 + kb$. Now $(ba)\bar{a} = (k-1)a\bar{a} = (k-1)[k\cdot 1 + k\cdot b]$, $b(a\bar{a}) = kb + kb^2$, and we have $b^2 = (k-1)\cdot 1 + (k-2)b$. Now Proposition 2.10 finishes the proof. ∎

PROPOSITION 3.7: *Let G be a non-cyclic finite group which contains a conjugacy class C such that*

(1) $\langle C\rangle = G$;

(2) $CC^{-1} = \{1\} \cup D$, $D \in \mathrm{Cla}(G)$ *and* $|D| \leq |C|$.

Then $G \simeq \mathbb{Z}_p^n : \langle c\rangle$ where the action of c on $\mathbb{Z}_p^n$ is defined by a Singer matrix. In particular, $G/Z(G) \simeq AGL_1(p^n)$.

Proof: The table algebra of conjugacy classes of G satisfies the conditions of Main Theorem 1.1 with $|C| = k \geq 3$. (If $|C| = 2$, then for any $g \in C$, $|G : C_G(g)| = 2$ implies that $C \subseteq C_G(g) < G$. This contradicts $\langle C\rangle = G$.) Let $N = \langle D\rangle$. If II(a) or (b) of Main Theorem 1.1 holds, then $|D| = |C|$, $G = Z(G)N$, and $|N| = k+1$ (if $\mathbf{B}_b \simeq_x \mathbf{T}_0(k)$) or $|N| = 2k+1$ (if $\mathbf{B}_b \simeq_x \mathbf{V}(3,k)$). For any $h \in D$, $|G : C_G(h)| = k$. Thus $|G : NC_G(h)| = k/|N : N \cap C_G(h)|$, where the denominator and numerator are relatively prime. Hence $N \leq C_G(h)$, so that N is abelian. Then G is abelian, which contradicts $|C| > 1$. So II(c) holds, $|D| = |C| - 1$, and $N = 1 \cup D$. Since G is transitive on $N^\#$, N is elementary abelian of order p^n for some p and exponent n.

In this case $C = cN$ for each $c \in C$. Now (1) implies that $G = \langle c, N\rangle = N\cdot\langle c\rangle$.

Since D is a conjugacy class of G and $N \leq C_G(D)$, $D = d^{\langle c\rangle}$ for each $d \in D$. Thus $\langle c\rangle$ acts transitively on the set D of cardinality $p^n - 1$. Hence the kernel of this action is a subgroup of $\langle c\rangle$ of index $p^n - 1$, or, equivalently, $\langle c^{p^n-1}\rangle \leq Z(G)$.

On the other hand, if $z \in Z(G)$ is an arbitrary element, then $z = c^i\cdot n$ for some $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Since $[Z(G), N] = 1$, $[c^i, N] = 1 \Rightarrow p^n - 1 | i \Rightarrow Z = (c^{p^h-1})^j \cdot n$.

But $c^{p^n-1} \in Z(G)$, hence $n \in Z(G)$. Now $N \cap Z(G) = 1$ implies that $z \in \langle c^{p^n-1} \rangle$. Thus $Z(G) = \langle c^{p^n-1} \rangle$. Since $N \cap \langle c \rangle \leq Z(G)$, $N \cap \langle c \rangle = \{1\}$.

Take an arbitrary $a \in C$. Let A be the matrix corresponding to the action of a on $N \simeq \mathbb{Z}_p^n$. Since $\langle a \rangle$ acts on $N \setminus \{1\}$ transitively, $\langle A \rangle$ acts on $\mathbb{Z}_p^n \setminus \{0\}$ transitively $\Rightarrow$ $\langle A \rangle$ is a Singer subgroup of $GL_n(p)$. The isomorphism $G \simeq \mathbb{Z}_p^n : \langle c \rangle$ follows from $G = N \cdot \langle c \rangle$ and $N \cap \langle c \rangle = \{1\}$. It follows from $G/Z(G) \simeq \mathbb{Z}_p^n : \langle A \rangle$ that $G/Z(G) \simeq AGL_1(p^n)$. ∎

3.3. Proof of Theorem 1.2.

Proof: Let $a \in \mathbf{B}^\#$, $|a| = k$, and $a\bar{a} = k \cdot 1 + \ell x$ for some $x \in \mathbf{B}^\#$, $\ell \in \mathbb{N}$ where $\ell|x| = k^2 - k$. Now $\langle a\bar{a}, x \rangle = \ell|x| = k^2 - k$. On the other hand, $\langle a, ax \rangle = \lambda_{axa} \cdot k$. Thus

$$ax = (k-1)a + \sum_{i=1}^{t} \alpha_i d_i \quad \text{where } d_i \in \mathbf{B}^\#. \tag{17}$$

Now since $|\mathsf{Supp}(x^2)| \leq 2$ and $|x| > k$, we can assume that $x^2 = |x| \cdot 1 + \gamma y$ where $y \in \mathbf{B}^\#$, $\gamma \in \mathbb{N}$. Therefore

$$\begin{aligned}\sum_{i=1}^{t} \alpha_i^2 |d_i| &= \langle ax, ax \rangle - (k-1)^2 |a| \\ &= \langle a\bar{a}, x\overline{x} \rangle - (k-1)^2 k = k|x| + (|x|-1)\ell|x|\delta_{xy} - (k-1)^2 k.\end{aligned}$$

Since $|x| = (k^2 - k)/\ell$, we have that either $\delta_{xy} \neq 0$ or

$$0 < \sum_{i=1}^{t} \alpha_i^2 |d_i| = k|x| - (k-1)^2 k = \left[\frac{k}{l} - (k-1)\right] k(k-1).$$

In the latter case, $l = 1$ and $\sum_{i=1}^{t} \alpha_i^2 |d_i| = (k-1)k$. But $\sum_{i=1}^{t} \alpha_i |d_i| = (k-1)^2 k > (k-1)k$, a contradiction.

Therefore, $x = y$. Then $\{1, x\}$ is a table subset and $\mathbf{B}_a = \mathbf{L}^x(\mathbf{B}_a)$, so Theorem 2.4 applies. ∎

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